



Generalizing the variational theory on time scales to include the delta indefinite integral

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ABSTRACT

We prove necessary optimality conditions of Euler–Lagrange type for generalized problems of the calculus of variations on time scales with a Lagrangian depending not only on the independent variable, an unknown function and its delta derivative, but also on a delta indefinite integral that depends on the unknown function. Such kinds of variational problems were considered by Euler himself and have been recently investigated in [J. Gregory, Generalizing variational theory to include the indefinite integral, higher derivatives, and a variety of means as cost variables, *Methods Appl. Anal.* 15 (4) (2008) 427–435]. Our results not only provide a generalization to previous results, but also give some other interesting optimality conditions as special cases.

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1. Introduction

In what follows, \mathbb{T} denotes a time scale with operators σ , ρ , μ , ν , Δ , and ∇ [1,2]. We also assume that there exist at least three points on the time scale: $a, b, s \in \mathbb{T}$ with $a < b < s$, and that the operator σ is delta differentiable. The main purpose of this paper is to generalize the Calculus of Variations on time scales (see [3–8] and the references therein) by considering the variational problem

$$\mathcal{L}(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t \longrightarrow \text{extr}, \quad (1)$$

where “extr” denotes “extremize” (i.e., minimize or maximize) and the variable z in the integrand is itself expressed in terms of an indefinite integral

$$z(t) = \int_a^t g(\tau, y^\sigma(\tau), y^\Delta(\tau)) \Delta \tau.$$

In Section 3.1 we obtain the Euler–Lagrange equation for problem (1) in the class of functions $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \quad (2)$$

for some fixed $\alpha, \beta \in \mathbb{R}$ (cf. Theorem 4). According to Fraser [9], the idea of generalizing the basic problem of the calculus of variations by considering a variational integral depending also on an indefinite integral (in the classical setting, that is, when $\mathbb{T} = \mathbb{R}$) was first considered by Euler in 1741. Our Euler–Lagrange equation is a generalization of the Euler–Lagrange equations obtained by Euler [9, Eq. (8)], Bohner [3], and Gregory [10]. The transversality conditions for problem (1) are

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obtained in Section 3.2. In Section 3.3 we prove a necessary optimality condition for the isoperimetric problem: problem (1)–(2) subject to the delta integral constraint

$$\mathcal{J}(y) = \int_a^b F(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t = \gamma$$

for some given $\gamma \in \mathbb{R}$. In Section 3.4 we explain how it is possible to prove backward versions of our results by means of Caputo's duality [11] (see also [12]). Finally, in Section 4 we provide some applications of our main results.

2. Preliminaries

For definitions, notations and results concerning the theory of time scales we refer the readers to the comprehensive books [1,2]. All the intervals in this paper are time scale intervals. Throughout the text we denote by $\partial_i f$ the partial derivative of a function f with respect to its i th argument.

We assume that

1. the admissible functions y belong to the class $C_{rd}^1(\mathbb{T}, \mathbb{R})$;
2. $(t, y, v, z) \rightarrow L(t, y, v, z)$ and $(t, y, v, z) \rightarrow F(t, y, v, z)$ have continuous partial derivatives with respect to y, v, z for all $t \in [a, b]$;
3. $(t, y, v) \rightarrow g(t, y, v)$ has continuous partial derivatives with respect to y, v for all $t \in [a, b]$;
4. $t \rightarrow L(t, y^\sigma(t), y^\Delta(t), z(t))$ and $t \rightarrow F(t, y^\sigma(t), y^\Delta(t), z(t))$ belong to the class $C_{rd}(\mathbb{T}, \mathbb{R})$ for any admissible function y ;
5. $t \rightarrow \partial_3 L(t, y^\sigma(t), y^\Delta(t), z(t))$, $t \rightarrow \partial_3 F(t, y^\sigma(t), y^\Delta(t), z(t))$ and $t \rightarrow \partial_3 g(t, y^\sigma(t), y^\Delta(t))$ belong to the class $C_{rd}^1(\mathbb{T}, \mathbb{R})$ for any admissible function y .

Definition 1. An admissible function $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ is said to be a local minimizer (resp. local maximizer) to problem (1)–(2) if there exists $\delta > 0$ such that $\mathcal{L}(y_*) \leq \mathcal{L}(y)$ (resp. $\mathcal{L}(y_*) \geq \mathcal{L}(y)$) for all admissible y satisfying the boundary conditions (2) and $\|y - y_*\| < \delta$, where

$$\|y\| = \sup_{t \in [a, b]^\kappa} |y^\sigma(t)| + \sup_{t \in [a, b]^\kappa} |y^\Delta(t)|.$$

Definition 2. We say that $\eta \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ is an admissible variation to problem (1)–(2) provided $\eta(a) = \eta(b) = 0$.

The following result, known as the *fundamental lemma of the calculus of variations on time scales*, is an important tool in the proofs of our main results. The proof of Lemma 3 follows immediately from [13, Theorem 15] and the duality arguments of Caputo [11].

Lemma 3. Let $f \in C_{rd}([a, b], \mathbb{R})$. Then

$$\int_a^b f(t) \eta^\sigma(t) \Delta t = 0 \quad \text{for all } \eta \in C_{rd}([a, b], \mathbb{R}) \text{ with } \eta(a) = \eta(b) = 0$$

if and only if $f(t) = 0$ for all $t \in [a, b]^\kappa$.

3. Main results

In order to simplify expressions, we introduce two operators, $[\cdot]$ and $\{\cdot\}$, defined in the following way:

$$[y](t) := (t, y^\sigma(t), y^\Delta(t), z(t)) \quad \text{and} \quad \{y\}(t) := (t, y^\sigma(t), y^\Delta(t)),$$

where $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$.

3.1. Euler–Lagrange equation

Theorem 4 (Necessary Optimality Condition to (1)–(2)). Suppose that y_* is a local minimizer or local maximizer to problem (1)–(2). Then y_* satisfies the Euler–Lagrange equation

$$\partial_2 L[y](t) - \frac{\Delta}{\Delta t} \partial_3 L[y](t) + \partial_2 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 L[y](\tau) \Delta \tau - \frac{\Delta}{\Delta t} \left(\partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 L[y](\tau) \Delta \tau \right) = 0 \quad (3)$$

for all $t \in [a, b]^\kappa$.

Proof. Suppose that y_* is a local minimizer (resp. maximizer) to problem (1)–(2). Let η be an admissible variation and define the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi(\epsilon) := \mathcal{L}(y_* + \epsilon \eta)$. It is clear that a necessary condition for y_* to be an extremizer is given by $\phi'(0) = 0$. Note that

$$\begin{aligned}\phi'(0) &= \int_a^b \left(\partial_2 L[y_*](t) \eta^\sigma(t) + \partial_3 L[y_*](t) \eta^\Delta(t) \right. \\ &\quad \left. + \partial_4 L[y_*](t) \cdot \int_a^t (\partial_2 g\{y_*\}(\tau) \eta^\sigma(\tau) + \partial_3 g\{y_*\}(\tau) \eta^\Delta(\tau)) \Delta\tau \right) \Delta t.\end{aligned}$$

Using the integration by parts formula, we obtain

$$\begin{aligned}\int_a^b \partial_3 L[y_*](t) \eta^\Delta(t) \Delta t &= [\partial_3 L[y_*](t) \eta(t)]_a^b - \int_a^b \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) \eta^\sigma(t) \Delta t, \\ \int_a^b \left(\partial_4 L[y_*](t) \cdot \int_a^t \partial_2 g\{y_*\}(\tau) \eta^\sigma(\tau) \Delta\tau \right) \Delta t &= \left[\int_b^t \partial_4 L[y_*](\tau) \Delta\tau \cdot \int_a^t \partial_2 g\{y_*\}(\tau) \eta^\sigma(\tau) \Delta\tau \right]_a^b \\ &\quad - \int_a^b \left(\int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \cdot \partial_2 g\{y_*\}(t) \eta^\sigma(t) \right) \Delta t \\ &= - \int_a^b \left(\partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \right) \eta^\sigma(t) \Delta t,\end{aligned}$$

and

$$\begin{aligned}\int_a^b \left(\partial_4 L[y_*](t) \cdot \int_a^t \partial_3 g\{y_*\}(\tau) \eta^\Delta(\tau) \Delta\tau \right) \Delta t &= \left[\int_b^t \partial_4 L[y_*](\tau) \Delta\tau \cdot \int_a^t \partial_3 g\{y_*\}(\tau) \eta^\Delta(\tau) \Delta\tau \right]_a^b \\ &\quad - \int_a^b \left(\int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \cdot \partial_3 g\{y_*\}(t) \eta^\Delta(t) \right) \Delta t \\ &= - \int_a^b \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \right) \eta^\Delta(t) \Delta t.\end{aligned}$$

Using again integration by parts in the last integral we obtain

$$\begin{aligned}\int_a^b \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \right) \eta^\Delta(t) \Delta t &= \left[\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \cdot \eta(t) \right]_a^b \\ &\quad - \int_a^b \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \right) \eta^\sigma(t) \Delta t.\end{aligned}$$

Since $\eta(a) = \eta(b) = 0$, then

$$\begin{aligned}\phi'(0) &= \int_a^b \left(\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \right) \right) \eta^\sigma(t) \Delta t.\end{aligned}$$

From the optimality condition $\phi'(0) = 0$ we conclude, by the fundamental lemma of the calculus of variations on time scales (Lemma 3), that

$$\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta\tau \right) = 0$$

for all $t \in [a, b]^\kappa$, proving the desired result. \square

Remark 5. Note that

1. The Euler–Lagrange equation (3) is a generalization of the Euler–Lagrange equation obtained by Euler in 1741 (if $\mathbb{T} = \mathbb{R}$, we obtain equation (8) of [9]).
2. Theorem 3.1 of [10] is a corollary of Theorem 4: choose $g(t, u, v) = u$ and consider the time scale to be the set of real numbers.
3. The Euler–Lagrange equation for the basic problem of the Calculus of Variations on time scales (see, e.g., [3]) is easily obtained from Theorem 4: in this case, $\partial_4 L = 0$ and therefore we get the equation

$$\partial_2 L(t, y^\sigma(t), y^\Delta(t)) - \frac{\Delta}{\Delta t} \partial_3 L(t, y^\sigma(t), y^\Delta(t)) = 0$$

for all $t \in [a, b]^\kappa$.

Remark 6. Theorem 4 gives the Euler–Lagrange equation in the delta-differential form. As in the classical case, one can obtain the Euler–Lagrange equation in the integral form. More precisely, the Euler–Lagrange equation in the delta integral form to problem (1)–(2) is

$$\partial_3 L[y](t) + \partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 L[y](\tau) \Delta \tau + \int_t^b \left(\partial_2 L[y](s) + \partial_2 g\{y\}(s) \cdot \int_{\sigma(s)}^b \partial_4 L[y](\tau) \Delta \tau \right) \Delta s = \text{const.}$$

3.2. Natural boundary conditions

We now consider the case when the values $y(a)$ and $y(b)$ are not necessarily specified.

Theorem 7 (Natural Boundary Conditions to (1)). Suppose that y_* is a local minimizer (resp. local maximizer) to problem (1). Then y_* satisfies the Euler–Lagrange equation (3). Moreover,

1. if $y(a)$ is free, then the natural boundary condition

$$\partial_3 L[y_*](a) = -\partial_3 g\{y_*\}(a) \cdot \int_{\sigma(a)}^b \partial_4 L[y_*](\tau) \Delta \tau \quad (4)$$

holds;

2. if $y(b)$ is free, then the natural boundary condition

$$\partial_3 L[y_*](b) = \partial_3 g\{y_*\}(b) \cdot \int_b^{\sigma(b)} \partial_4 L[y_*](\tau) \Delta \tau \quad (5)$$

holds.

Proof. Suppose that y_* is a local minimizer (resp. maximizer) to problem (1). Let $\eta \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ and define the function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ by $\phi(\epsilon) := \mathcal{L}(y_* + \epsilon \eta)$. It is clear that a necessary condition for y_* to be an extremizer is given by $\phi'(0) = 0$. From the arbitrariness of η , and using similar arguments as the ones used in the proof of Theorem 4, we conclude that y_* satisfies the Euler–Lagrange equation (3).

1. Suppose now that $y(a)$ is free. If $y(b) = \beta$ is given, then $\eta(b) = 0$; if $y(b)$ is free, then we restrict ourselves to those η for which $\eta(b) = 0$. Therefore,

$$\begin{aligned} 0 &= \phi'(0) \\ &= \int_a^b \left(\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \right) \eta^\sigma(t) \Delta t - \partial_3 L[y_*](a) \cdot \eta(a) \\ &\quad + \partial_3 g\{y_*\}(a) \cdot \int_b^{\sigma(a)} \partial_4 L[y_*](\tau) \Delta \tau \cdot \eta(a). \end{aligned} \quad (6)$$

Using the Euler–Lagrange equation (3) into (6) we obtain

$$\left(-\partial_3 L[y_*](a) + \partial_3 g\{y_*\}(a) \cdot \int_b^{\sigma(a)} \partial_4 L[y_*](\tau) \Delta \tau \right) \cdot \eta(a) = 0.$$

From the arbitrariness of η it follows that

$$\partial_3 L[y_*](a) = \partial_3 g\{y_*\}(a) \cdot \int_b^{\sigma(a)} \partial_4 L[y_*](\tau) \Delta \tau.$$

2. Suppose now that $y(b)$ is free. If $y(a) = \alpha$, then $\eta(a) = 0$; if $y(a)$ is free, then we restrict ourselves to those η for which $\eta(a) = 0$. Thus,

$$\begin{aligned} 0 &= \phi'(0) \\ &= \int_a^b \left(\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \right) \eta^\sigma(t) \Delta t \\ &\quad + \left(\partial_3 L[y_*](b) - \partial_3 g\{y_*\}(b) \cdot \int_b^{\sigma(b)} \partial_4 L[y_*](\tau) \Delta \tau \right) \cdot \eta(b). \end{aligned} \quad (7)$$

Using the Euler–Lagrange equation (3) into (7), and from the arbitrariness of η , it follows that

$$\partial_3 L[y_*(b)] = \partial_3 g\{y_*(b)\} \cdot \int_b^{\sigma(b)} \partial_4 L[y_*(\tau)] \Delta \tau. \quad \square$$

Remark 8. In the classical setting, $\mathbb{T} = \mathbb{R}$ and L does not depend on z . Then, Eqs. (4) and (5) reduce to the well-known natural boundary conditions

$$\partial_3 L(a, y_*(a), y'_*(a)) = 0 \quad \text{and} \quad \partial_3 L(b, y_*(b), y'_*(b)) = 0,$$

respectively.

3.3. Isoperimetric problem

We now study the isoperimetric problem on time scales with a delta integral constraint, both for normal and abnormal extremizers. The problem consists of minimizing or maximizing the functional

$$\mathcal{L}(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t, \quad (8)$$

where the variable z in the integrand is itself expressed in terms of an indefinite delta integral

$$z(t) = \int_a^t g(\tau, y^\sigma(\tau), y^\Delta(\tau)) \Delta \tau,$$

in the class of functions $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$, satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \quad (9)$$

and the delta integral constraint

$$\mathcal{J}(y) = \int_a^b F(t, y^\sigma(t), y^\Delta(t), z(t)) \Delta t = \gamma \quad (10)$$

for some given $\alpha, \beta, \gamma \in \mathbb{R}$.

Definition 9. We say that $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ is a local minimizer (resp. local maximizer) to the isoperimetric problem (8)–(10) if there exists $\delta > 0$ such that $\mathcal{L}(y_*) \leq \mathcal{L}(y)$ (resp. $\mathcal{L}(y_*) \geq \mathcal{L}(y)$) for all admissible y satisfying the boundary conditions (9), the isoperimetric constraint (10), and $\|y - y_*\| < \delta$.

Definition 10. We say that $y \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ is an extremal to \mathcal{J} if y satisfies the Euler–Lagrange equation (3) relatively to \mathcal{J} . An extremizer (i.e., a local minimizer or a local maximizer) to problem (8)–(10) that is not an extremal to \mathcal{J} is said to be a normal extremizer; otherwise (i.e., if it is an extremal to \mathcal{J}), the extremizer is said to be abnormal.

Theorem 11 (Necessary Optimality Condition for Normal Extremizers of (8)–(10)). Suppose that $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ gives a local minimum or a local maximum to the functional \mathcal{L} subject to the boundary conditions (9) and the integral constraint (10). If y_* is not an extremal to \mathcal{J} , then there exists a real λ such that y_* satisfies the equation

$$\partial_2 H[y](t) - \frac{\Delta}{\Delta t} \partial_3 H[y](t) + \partial_2 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau - \frac{\Delta}{\Delta t} \left(\partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau \right) = 0 \quad (11)$$

for all $t \in [a, b]^\kappa$, where $H = L - \lambda F$.

Proof. Suppose that $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ is a normal extremizer to problem (8)–(10). Define the real functions $\phi, \psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$\phi(\epsilon_1, \epsilon_2) = \mathcal{L}(y_* + \epsilon_1 \eta_1 + \epsilon_2 \eta_2),$$

$$\psi(\epsilon_1, \epsilon_2) = \mathcal{J}(y_* + \epsilon_1 \eta_1 + \epsilon_2 \eta_2) - \gamma,$$

where η_2 is a fixed variation (that we will choose later) and η_1 is an arbitrary variation. Note that

$$\begin{aligned} \frac{\partial \psi}{\partial \epsilon_2}(0, 0) &= \int_a^b \left(\partial_2 F[y_*](t) \eta_2^\sigma(t) + \partial_3 F[y_*](t) \eta_2^\Delta(t) \right. \\ &\quad \left. + \partial_4 F[y_*](t) \cdot \int_a^t (\partial_2 g\{y_*\}(\tau) \eta_2^\sigma(\tau) + \partial_3 g\{y_*\}(\tau) \eta_2^\Delta(\tau)) \Delta \tau \right) \Delta t. \end{aligned}$$

Integration by parts and $\eta_2(a) = \eta_2(b) = 0$ gives

$$\begin{aligned} \frac{\partial \psi}{\partial \epsilon_2}(0, 0) &= \int_a^b \left(\partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right) \right) \eta_2^\sigma(t) \Delta t. \end{aligned}$$

Since, by hypothesis, y_* is not an extremal to \mathcal{J} , then we can choose η_2 such that $\frac{\partial \psi}{\partial \epsilon_2}(0, 0) \neq 0$. We keep η_2 fixed. Since $\psi(0, 0) = 0$, by the implicit function theorem there exists a function h , defined in a neighborhood V of zero, such that $h(0) = 0$ and $\psi(\epsilon_1, h(\epsilon_1)) = 0$ for any $\epsilon_1 \in V$, i.e., there exists a subset of variation curves $y = y_* + \epsilon_1 \eta_1 + h(\epsilon_1) \eta_2$ satisfying the isoperimetric constraint. Note that $(0, 0)$ is an extremizer of ϕ subject to the constraint $\psi = 0$ and

$$\nabla \psi(0, 0) \neq (0, 0).$$

By the Lagrange multiplier rule (cf., e.g., [14]), there exists some constant $\lambda \in \mathbb{R}$ such that

$$\nabla \phi(0, 0) = \lambda \nabla \psi(0, 0). \quad (12)$$

Since

$$\begin{aligned} \frac{\partial \phi}{\partial \epsilon_1}(0, 0) &= \int_a^b \left(\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \right) \eta_1^\sigma(t) \Delta t \end{aligned}$$

and

$$\begin{aligned} \frac{\partial \psi}{\partial \epsilon_1}(0, 0) &= \int_a^b \left(\partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right) \right) \eta_1^\sigma(t) \Delta t, \end{aligned}$$

it follows from (12) that

$$\begin{aligned} 0 &= \int_a^b \left(\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \right. \\ &\quad \left. - \lambda \left(\partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \right. \\ &\quad \left. \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right) \right) \right) \eta_1^\sigma(t) \Delta t. \end{aligned}$$

Using the fundamental lemma of the calculus of variations (Lemma 3), and recalling that η_1 is arbitrary, we conclude that

$$\begin{aligned} 0 &= \partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \\ &\quad - \lambda \left(\partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \\ &\quad \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right) \right) \end{aligned}$$

for all $t \in [a, b]^\kappa$, proving that $H = L - \lambda F$ satisfies the Euler–Lagrange equation (11). \square

Theorem 12 (Necessary Optimality Condition for Normal and Abnormal Extremizers of (8)–(10)). Suppose that $y_* \in C_{rd}^1(\mathbb{T}, \mathbb{R})$ gives a local minimum or a local maximum to the functional \mathcal{L} subject to the boundary conditions (9) and the integral constraint (10). Then there exist two constants λ_0 and λ , not both zero, such that y_* satisfies the equation

$$\partial_2 H[y](t) - \frac{\Delta}{\Delta t} \partial_3 H[y](t) + \partial_2 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau - \frac{\Delta}{\Delta t} \left(\partial_3 g\{y\}(t) \cdot \int_{\sigma(t)}^b \partial_4 H[y](\tau) \Delta \tau \right) = 0 \quad (13)$$

for all $t \in [a, b]^\kappa$, where $H = \lambda_0 L - \lambda F$.

Proof. Following the proof of Theorem 11, since $(0, 0)$ is an extremizer of ϕ subject to the constraint $\psi = 0$, the abnormal Lagrange multiplier rule (cf., e.g., [14]) guarantees the existence of two reals λ_0 and λ , not both zero, such that

$$\lambda_0 \nabla \phi = \lambda \nabla \psi.$$

Therefore,

$$\lambda_0 \frac{\partial \phi}{\partial \epsilon_1}(0, 0) = \lambda \frac{\partial \psi}{\partial \epsilon_1}(0, 0)$$

and hence,

$$\begin{aligned} 0 &= \int_a^b \left(\lambda_0 \left(\partial_2 L[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 L[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right. \right. \\ &\quad \left. \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 L[y_*](\tau) \Delta \tau \right) \right) \right. \\ &\quad \left. - \lambda \left(\partial_2 F[y_*](t) - \frac{\Delta}{\Delta t} \partial_3 F[y_*](t) - \partial_2 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right. \right. \\ &\quad \left. \left. + \frac{\Delta}{\Delta t} \left(\partial_3 g\{y_*\}(t) \cdot \int_b^{\sigma(t)} \partial_4 F[y_*](\tau) \Delta \tau \right) \right) \right) \eta_1^\sigma(t) \Delta t. \end{aligned}$$

From the arbitrariness of η_1 and [Lemma 3](#) it is clear that Eq. (13) holds for all $t \in [a, b]^\kappa$, where $H = \lambda_0 L - \lambda F$. \square

Remark 13. Note that

1. If y_* is a normal extremizer, then one can consider, by [Theorem 11](#), $\lambda_0 = 1$ in [Theorem 12](#). The condition $(\lambda_0, \lambda) \neq (0, 0)$ guarantees that [Theorem 12](#) is a useful necessary condition.
2. Theorem 3.4 of [15] is a corollary of our [Theorem 11](#): in that case, $\partial_4 H = 0$ and we simply obtain

$$\partial_2 H(t, y^\sigma(t), y^\Delta(t)) - \frac{\Delta}{\Delta t} \partial_3 H(t, y^\sigma(t), y^\Delta(t)) = 0$$

for all $t \in [a, b]^\kappa$.

We present two important corollaries that are obtained from [Theorem 12](#) choosing the time scale to be $\mathbb{T} = h\mathbb{Z} := \{hz : z \in \mathbb{Z}\}$, $h > 0$, and $\mathbb{T} = q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$, $q > 1$. In what follows we use the standard notation of *quantum calculus* (see, e.g., [16–18]):

$$\Delta_h y(t) := \frac{y(t+h) - y(t)}{h} \quad \text{and} \quad D_q y(t) := \frac{y(qt) - y(t)}{(q-1)t}.$$

Corollary 14. Let $h > 0$ and suppose that y_* is a solution to the discrete-time problem

$$\mathcal{L}(y) = \sum_{t=a}^{b-h} L(t, y(t+h), \Delta_h y(t), z(t)) \longrightarrow \text{extr}$$

with

$$z(t) = \sum_{\tau=a}^{t-h} g(\tau, y(\tau+h), \Delta_h y(\tau))$$

in the class of functions y satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

and the constraint

$$\mathcal{J}(y) = \sum_{t=a}^{b-h} F(t, y(t+h), \Delta_h y(t), z(t)) = \gamma$$

for some given $\alpha, \beta, \gamma \in \mathbb{R}$. Then there exist two constants λ_0 and λ , not both zero, such that

$$\begin{aligned} 0 &= \partial_2 H(t, y_*(t+h), \Delta_h y_*(t), z_*(t)) - \Delta_h \partial_3 H(t, y_*(t+h), \Delta_h y_*(t), z_*(t)) \\ &\quad + \partial_2 g(t, y_*(t+h), \Delta_h y_*(t)) \cdot \sum_{\tau=t+h}^{b-h} \partial_4 H(\tau, y_*(\tau+h), \Delta_h y_*(\tau), z_*(\tau)) \\ &\quad - \Delta_h \left(\partial_3 g(t, y_*(t+h), \Delta_h y_*(t)) \cdot \sum_{\tau=t+h}^{b-h} \partial_4 H(\tau, y_*(\tau+h), \Delta_h y_*(\tau), z_*(\tau)) \right) \end{aligned}$$

for all $t \in \{a, a+h, \dots, b-h\}$, where $H = \lambda_0 L - \lambda F$.

Proof. Choose $\mathbb{T} = h\mathbb{Z}$, where $a, b \in \mathbb{T}$. The result follows from [Theorem 12](#). \square

Corollary 15. Let $q > 1$ and suppose that y_* is a solution to the quantum problem

$$\mathcal{L}(y) = \sum_{t=a}^{bq^{-1}} L(t, y(qt), D_q y(t), z(t)) \longrightarrow \text{extr}$$

with

$$z(t) = \sum_{\tau=a}^{tq^{-1}} g(\tau, y(q\tau), D_q y(\tau))$$

in the class of functions y satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta$$

and the constraint

$$\mathcal{J}(y) = \sum_{t=a}^{bq^{-1}} F(t, y(qt), D_q y(t), z(t)) = \gamma$$

for some given $\alpha, \beta, \gamma \in \mathbb{R}$. Then there exist two constants λ_0 and λ , not both zero, such that

$$\begin{aligned} 0 = & \partial_2 H(t, y_*(qt), D_q y_*(t), z_*(t)) - D_q \partial_3 H(t, y_*(qt), D_q y_*(t), z_*(t)) \\ & + \partial_2 g(t, y_*(qt), D_q y_*(t)) \cdot \sum_{\tau=qt}^{bq^{-1}} \partial_4 H(\tau, y_*(q\tau), D_q y_*(\tau), z_*(\tau)) \\ & - D_q \left(\partial_3 g(t, y_*(qt), D_q y_*(t)) \cdot \sum_{\tau=qt}^{bq^{-1}} \partial_4 H(\tau, y_*(q\tau), D_q y_*(\tau), z_*(\tau)) \right) \end{aligned}$$

for all $t \in \{a, qa, \dots, bq^{-1}\}$, where $H = \lambda_0 L - \lambda F$.

Proof. Choose $\mathbb{T} = q^{\mathbb{N}_0}$, where $a, b \in \mathbb{T}$. The result follows from [Theorem 12](#). \square

3.4. Duality

In the paper [11] (see also [19,20]) Caputo states that the *delta calculus* and the *nabla calculus* on time scales are the “dual” of each other. A *Duality Principle* is presented, that basically asserts that it is possible to obtain results for the *nabla calculus* directly from results on the *delta calculus* and vice versa. Using the duality arguments of Caputo it is possible to prove easily the *nabla* versions of [Theorems 4, 7, 11](#) and [12](#).

In what follows we assume that there exist at least three points on the time scale: $r, a, b \in \mathbb{T}$ with $r < a < b$, and that the operator ρ is nabla differentiable. The following theorem is the *nabla* version of [Theorem 12](#), where the variational problem consists of minimizing or maximizing the functional

$$\mathcal{L}(y) = \int_a^b L(t, y^\rho(t), y^\nabla(t), z(t)) \nabla t, \tag{14}$$

the variable z in the integrand being itself expressed in terms of a nabla indefinite integral

$$z(t) = \int_a^t g(\tau, y^\rho(\tau), y^\nabla(\tau)) \nabla \tau,$$

in the class of functions $y \in C_{ld}^1(\mathbb{T}, \mathbb{R})$ satisfying the boundary conditions

$$y(a) = \alpha \quad \text{and} \quad y(b) = \beta \tag{15}$$

and the nabla integral constraint

$$\mathcal{J}(y) = \int_a^b F(t, y^\rho(t), y^\nabla(t), z(t)) \nabla t = \gamma \tag{16}$$

for some given $\alpha, \beta, \gamma \in \mathbb{R}$. We assume that

1. the admissible functions y belong to the class $C_{ld}^1(\mathbb{T}, \mathbb{R})$;

2. $(t, y, v, z) \rightarrow L(t, y, v, z)$ and $(t, y, v, z) \rightarrow F(t, y, v, z)$ have continuous partial derivatives with respect to y, v, z for all $t \in [a, b]$;
3. $(t, y, v) \rightarrow g(t, y, v)$ has continuous partial derivatives with respect to y, v for all $t \in [a, b]$;
4. $t \rightarrow L(t, y^\rho(t), y^\nabla(t), z(t))$ and $t \rightarrow F(t, y^\rho(t), y^\nabla(t), z(t))$ belong to the class $C_{ld}(\mathbb{T}, \mathbb{R})$ for any admissible function y ;
5. $t \rightarrow \partial_3 L(t, y^\rho(t), y^\nabla(t), z(t)), t \rightarrow \partial_3 F(t, y^\rho(t), y^\nabla(t), z(t))$ and $t \rightarrow \partial_3 g(t, y^\rho(t), y^\nabla(t))$ belong to the class $C_{ld}^1(\mathbb{T}, \mathbb{R})$ for any admissible function y .

The following operators are used:

$$[y](t) := (t, y^\rho(t), y^\nabla(t), z(t)) \quad \text{and} \quad \langle y \rangle(t) := (t, y^\rho(t), y^\nabla(t)), \quad \text{where } y \in C_{ld}^1(\mathbb{T}, \mathbb{R}).$$

Theorem 16 (Necessary Optimality Condition for Normal and Abnormal Extremizers of (14)–(16)). Suppose that $y_* \in C_{ld}^1(\mathbb{T}, \mathbb{R})$ gives a local minimum or a local maximum to the functional \mathcal{L} subject to the boundary conditions (15) and the integral constraint (16). Then there exist two constants λ_0 and λ , not both zero, such that y_* satisfies the equation

$$\partial_2 H[y](t) - \frac{\nabla}{\nabla t} \partial_3 H[y](t) + \partial_2 g \langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left(\partial_3 g \langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau \right) = 0$$

for all $t \in [a, b]_\kappa$, where $H = \lambda_0 L - \lambda F$.

Remark 17. Theorem 2 of [21] is a corollary of our Theorem 16: in that case $\partial_4 H = 0$, and one obtains

$$\partial_2 H(t, y^\rho(t), y^\nabla(t)) - \frac{\nabla}{\nabla t} \partial_3 H(t, y^\rho(t), y^\nabla(t)) = 0$$

for all $t \in [a, b]_\kappa$.

From Theorem 4, via duality, one can easily obtain the Euler–Lagrange equation for the nabla problem (14)–(15) (or from Theorem 16 noting that, since there is no nabla integral constraint, $F = 0$ and $\gamma = 0$).

Theorem 18 (Necessary Optimality Condition to (14)–(15)). Suppose that y_* is a local minimizer or local maximizer to problem (14)–(15). Then y_* satisfies the Euler–Lagrange equation

$$\partial_2 L[y](t) - \frac{\nabla}{\nabla t} \partial_3 L[y](t) + \partial_2 g \langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 L[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left(\partial_3 g \langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 L[y](\tau) \nabla \tau \right) = 0 \quad (17)$$

for all $t \in [a, b]_\kappa$.

Remark 19. As a corollary of Theorem 18 we obtain the Euler–Lagrange equation for the basic problem of the calculus of variations on nabla calculus [13] (see also [22]). In that case $\partial L_4 = 0$ and one obtains that

$$\partial_2 L(t, y^\rho(t), y^\nabla(t)) - \frac{\nabla}{\nabla t} \partial_3 L(t, y^\rho(t), y^\nabla(t)) = 0$$

for all $t \in [a, b]_\kappa$.

Remark 20. Theorem 18 gives the Euler–Lagrange equation in the nabla-differential form. The Euler–Lagrange equation in the nabla integral form to problem (14)–(15) is

$$\partial_3 L[y](t) + \partial_3 g \langle y \rangle(t) \cdot \int_{\rho(t)}^b \partial_4 L[y](\tau) \Delta \tau + \int_t^b \left(\partial_2 L[y](s) + \partial_2 g \langle y \rangle(s) \cdot \int_{\rho(s)}^b \partial_4 L[y](\tau) \Delta \tau \right) \Delta s = \text{const.}$$

Applying the duality arguments of Caputo to Theorem 7 the following result is obtained.

Theorem 21 (Natural Boundary Conditions to (14)). Suppose that y_* is a local minimizer (resp. local maximizer) to problem (14). Then y_* satisfies the Euler–Lagrange equation (17). Moreover,

1. if $y(a)$ is free, then the natural boundary condition

$$\partial_3 L[y_*](a) = -\partial_3 g \langle y_* \rangle(a) \cdot \int_{\rho(a)}^b \partial_4 L[y_*](\tau) \Delta \tau$$

holds;

2. if $y(b)$ is free, then the natural boundary condition

$$\partial_3 L[y_*](b) = -\partial_3 g \langle y_* \rangle(b) \cdot \int_{\rho(b)}^b \partial_4 L[y_*](\tau) \Delta \tau$$

holds.

4. Applications

From now on we assume that \mathbb{T} satisfies the following condition (H):

(H) for each $t \in \mathbb{T}$, $\rho(t) = a_1 t + a_0$ for some $a_1 \in \mathbb{R}^+$ and $a_0 \in \mathbb{R}$.

Remark 22. Note that condition (H) implies that ρ is nabla differentiable and $\rho^\nabla(t) = a_1$, $t \in \mathbb{T}_\kappa$. Also note that condition (H) englobes the differential calculus ($\mathbb{T} = \mathbb{R}$, $a_1 = 1$, $a_0 = 0$), the difference calculus ($\mathbb{T} = \mathbb{Z}$, $a_1 = 1$, $a_0 = -1$), the h -calculus ($\mathbb{T} = h\mathbb{Z}$, for some $h > 0$, $a_1 = 1$, $a_0 = -h$), and the q -calculus ($\mathbb{T} = q^{\mathbb{N}_0}$ for some $q > 1$, $a_1 = \frac{1}{q}$, $a_0 = 0$).

The following result illustrates an application of [Theorem 16](#).

Proposition 23. Suppose that \mathbb{T} satisfies condition (H), ξ is a real parameter, and $k \in \mathbb{R}$ is a given constant. Suppose that $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a C^2 function that satisfies the conditions:

- (A1) $\partial_1 f(y^\rho(t), \xi) \neq -ka_1$ for all t in some non-degenerate interval $I \subseteq [a, b]$, for all ξ and for all admissible function y ;
 (A2) $\partial_{1,1}^2 f(y^\rho(t), \xi) \neq 0$ for all t in some non-degenerate interval $I \subseteq [a, b]$, for all ξ and for all admissible function y .

Consider

$$L(t, y, v, z) = f(y, \xi) + kz, \quad g(t, y, v) = v \quad \text{and} \quad F(t, y, v, z) = y.$$

If y_* is a solution to problem (14)–(16), then $y_*(t) = \alpha$, $t \in [a, b]_\kappa$.

Proof. Suppose that y_* is an extremizer to problem (14)–(16). By [Theorem 16](#) there exist two constants λ_0 and λ , not both zero, such that y_* satisfies the equation

$$\partial_2 H[y](t) - \frac{\nabla}{\nabla t} \partial_3 H[y](t) + \partial_2 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left(\partial_3 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau \right) = 0 \quad (18)$$

for all $t \in [a, b]_\kappa$, where $H = \lambda_0 L - \lambda F$. Since

$$\partial_2 H = \lambda_0 \partial_1 f - \lambda, \quad \partial_3 H = 0, \quad \partial_4 H = \lambda_0 k, \quad \partial_2 g = 0 \quad \text{and} \quad \partial_3 g = 1,$$

then Eq. (18) reduces to

$$\lambda_0 (\partial_1 f(y_*^\rho(t), \xi) + ka_1) = \lambda, \quad t \in [a, b]_\kappa. \quad (19)$$

Note that if $\lambda_0 = 0$, then $\lambda = 0$ violates the condition that λ_0 and λ do not vanish simultaneously. If $\lambda = 0$, then Eq. (19) reduces to $\lambda_0 (\partial_1 f(y_*^\rho(t), \xi) + ka_1) = 0$. By assumption (A1) we conclude that $\lambda_0 = 0$, which again contradicts the fact that λ_0 and λ are not both zero. Consequently, we can assume, without loss of generality, that $\lambda_0 = 1$. Hence, Eq. (19) takes the form

$$\partial_1 f(y_*^\rho(t), \xi) = \lambda - ka_1, \quad t \in [a, b]_\kappa.$$

By assumption (A2) we conclude that

$$y_*^\rho(t) = \text{const}, \quad t \in [a, b]_\kappa.$$

Since $y(a) = \alpha$, we obtain that $y_*(t) = \alpha$ for any $t \in [a, b]_\kappa$. \square

Observe that the solution to the class of problems considered in [Proposition 23](#) is a constant function that depends only on the boundary conditions (and the isoperimetric constraint) but not explicitly on the integrand function and its parameters.

Remark 24. By the isoperimetric constraint (16), a necessary condition for the problem of [Proposition 23](#) to have a solution is that $\alpha = \frac{\gamma}{b-a}$.

Remark 25. Let b be a left dense point. Then, by the boundary conditions (15), a necessary condition for the problem of [Proposition 23](#) to have solution is that $\alpha = \beta$.

Remark 26 (Cf. [23]). Let $\mathbb{T} = \mathbb{R}$. Suppose that $\alpha = \frac{\gamma}{b-a} = \beta$.

1. If $\partial_{1,1}^2 f(y(t), \xi) > 0$ for all $t \in [a, b]$, for all ξ and for all admissible function y , then problem (14)–(16) has a unique minimizer.
2. If $\partial_{1,1}^2 f(y(t), \xi) < 0$ for all $t \in [a, b]$, for all ξ and for all admissible function y , then problem (14)–(16) has a unique maximizer.

We end the paper with an example of application of the nabla version of [Theorem 11](#).

Example 27. Let $q : [a, b] \rightarrow \mathbb{R}$ be a continuous function and $y^{\nabla^2} := (y^{\nabla})^{\nabla}$. Suppose that $y_* \in C_{ld}^2$ is an extremizer for

$$\mathcal{L}(y) = \int_a^b \left((y^{\nabla})^2(t) - q(t)(y^{\rho})^2(t) + 2 \int_a^t y^{\nabla}(\tau) \nabla \tau \right) \nabla t$$

subject to the boundary conditions

$$y(a) = 0 \quad \text{and} \quad y(b) = 0$$

and the nabla integral constraint

$$\mathcal{J}(y) = \int_a^b (y^{\rho})^2(t) \nabla t = 1. \quad (20)$$

Note that any extremal to \mathcal{J} does not satisfy the isoperimetric constraint (20). Hence, this problem has no abnormal extremizers and, by the nabla version of Theorem 11, there exists $\lambda \in \mathbb{R}$ such that y_* satisfies the equation

$$\partial_2 H[y](t) - \frac{\nabla}{\nabla t} \partial_3 H[y](t) + \partial_2 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau - \frac{\nabla}{\nabla t} \left(\partial_3 g(y)(t) \cdot \int_{\rho(t)}^b \partial_4 H[y](\tau) \nabla \tau \right) = 0 \quad (21)$$

for all $t \in [a, b]_k$, where $H = L - \lambda F$ and

$$L(t, y, v, z) = v^2 - q(t)y^2 + 2z, \quad g(t, y, v) = v, \quad \text{and} \quad F(t, y, v, z) = y^2.$$

Since

$$\partial_2 H = -2qy - 2\lambda y, \quad \partial_3 H = 2v, \quad \partial_4 H = 2, \quad \partial_2 g = 0, \quad \text{and} \quad \partial_3 g = 1,$$

then Eq. (21) reduces to

$$y^{\nabla^2}(t) + q(t)y^{\rho}(t) + \lambda y^{\rho}(t) = \partial_4 H \cdot \frac{a_1}{2}, \quad t \in [a, b]_{k^2}. \quad (22)$$

Note that in the basic problem of calculus of variations on time scales, $\partial_4 H = 0$, and we obtain the nabla version of the well-known Sturm–Liouville eigenvalue equation:

$$y^{\nabla^2}(t) + q(t)y^{\rho}(t) + \lambda y^{\rho}(t) = 0, \quad t \in [a, b]_{k^2}$$

(see [24,15]). The study of solutions to Eq. (22) in the case $\partial_4 H \neq 0$ is an interesting open problem.

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